PROBABILITY

1.3 INTRODUCTION

The theory of mathematical probability has its origin in the 17th century. There are three different approaches of measuring probabilities. They are classical probability, relative frequency of occurrence and axiomatic probability. In this chapter, we shall study the classical probability and the axiomatic approach.

1.3.1 Definition of Probability (Mathematics) :

The outcomes of a random experiment are termed as events. The probability for the occurrence of an event A is defined as the ratio between the number of favourable outcomes fro the occurrence of the event and the total number of possible outcomes, i.e.,

$$\text{Probability of an event} = \frac{\text{Number of favourable events}}{\text{Total number of outcomes}}$$

Example : 1 ***Suppose a coin is tossed. There are two possible outcomes, head and tail. Both are equally likely events.***

The probability of getting head $= \frac{1}{2}$

Definition of Probability (Axiomatic)

Let S' be a sample space and A an event in S. Then P(A) is called the probability of the event of the following conditions are satisfied.

(i) $P(A) \geq 0$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events $P(A \cup B) = P(A) + P(B)$

1.3.2 SIMPLE THEOREMS :

Theorem 1 :

Prove that $P(\bar{A}) = 1 - P(A)$

Proof :

Since $\bar{A}$ is the complement of the event A.

$S = A \cup \bar{A}$

$$P(S) = P(A \cup \bar{A})$$

$$1 = P(A \cup \bar{A}) \quad \rightarrow \text{using Axiom (i)}$$

$$1 = P(A) + P(\bar{A}) \quad \rightarrow \text{using Axiom (iii)}$$

$$1 - P(A) = P(\bar{A})$$

i.e $\boxed{P(\bar{A}) = 1 - P(A)}$

Theorem 2 :

Probability of an impossible event is zero.

i.e., $P(\phi) = 0$

Proof :

The sample space 'S' and the impossible event ϕ are mutually exclusive.

i.e., $S \cup \phi = S$

$$P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S)$$

$$P(\phi) = 0$$

1.3.3 Trial and Event

Consider an experiment of throwing a coin. When tossing a coin, we may get a ***head (H) or tail (T)***. Here tossing of a coin is a trial and getting a head or tail is an event.

Eg : Throwing of a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Mutually Exclusive Events [Disjoint Events] :

Two events are said to be mutually exclusive when the occurrence of one does not affects the occurrence of the other.

In other words, if A and B are mutually exclusive events and if A happens the B will not happen.

Example :

In the throw of a single dice, the events of getting 1, 2, 3, ... 6 are mutually exclusive.

Exhaustive Events :

Events are said to be exhaustive when they include all possibilities.

Conditional Probability :

The conditional probability of event A, when the event B has already happened is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

(or)

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

Independent Events :

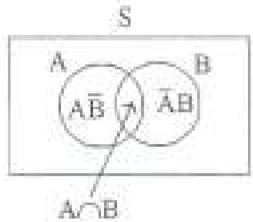
Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other.

1.4 LAWS OF PROBABILITY

Theorem : 1 *Addition Theorem*

If A and B are any two events and are not disjoint.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Proof :

$$A \cup B = A \cup \bar{A}B$$

$$P(A \cup B) = P(A \cup \bar{A}B)$$

$$P(A \cup B) = P(A) + P(\bar{A}B) \qquad \ldots (1)$$

But $$B = AB \cup \bar{A}B$$

$$P(B) = P(AB \cup \bar{A}B)$$

$$= P(AB) + P(\bar{A}B)$$

$$P(B) - P(AB) = P(\bar{A}B) \qquad \ldots (2)$$

Substitute (2) in (1)

$$\boxed{P(A \cup B) = P(A) + P(B) - P(AB)}$$

Note : If A and B are independent events the P(AB) = P(A).P(B).

Theorem : 2

If A and B are independents, prove that

(i) A and B are independent i.e., $P(\bar{A}\bar{B}) = P(\bar{A})\ P(\bar{B})$

(ii) A and B are independent i.e., $P(\bar{A}B) = P(\bar{A})\ P(B)$

(iii) A and B are independent i.e., $P(A\bar{B}) = P(A)\ P(\bar{B})$

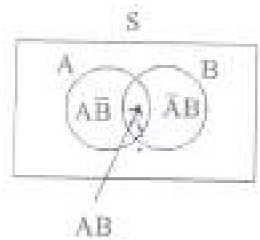
(i) $P(\bar{A}\bar{B}) = P(\bar{A}) \,.\, P(\bar{B})$

Proof

$$
\begin{aligned}
P(\overline{A \cup B}) &= 1 - P(A \cup B) \\
&= 1 - [P(A) + P(B) - P(AB)] \\
&= 1 - P(A) - P(B) + P(A).P(B) \\
&= P(\bar{A}) - P(B)\,[1 - P(A)] \\
&= P(\bar{A}) - P(B).P(A) \\
&= P(\bar{A})\,[1 - P(B)] \\
P(\overline{A \cup B}) &= P(\bar{A}).\,P(\bar{B}) \\
P(\overline{AB}) &= P(\bar{A}).P(B) \text{ using De Margon's law}
\end{aligned}
$$

(ii) $P(\bar{A}B) = P(\bar{A}) \,.\, P(B)$

Proof



$$
\begin{aligned}
B &= AB \cup \bar{A}B \\
P(B) &= P(AB \cup \bar{A}B)] \\
&= P(AB) + P(\bar{A}B) \\
P(B) - P(AB) &= P(\bar{A}B) \\
P(B) - P(A)\,.\,P(B) &= P(\bar{A}B) \\
P(B)\,[1 - P(A)] &= P(\bar{A}B) \\
P(B).P(\bar{A}) &= P(\bar{A}B)
\end{aligned}
$$

i.e $\boxed{P(\bar{A}B) = P(\bar{A}).P(B)}$

(iii) $P(A\bar{B}) = P(A) . P(\bar{B})$

Proof

$$A = AB \cup A\bar{B}$$

$$P(A) = P(AB \cup A\bar{B})]$$

$$= P(AB) + P(A\bar{B})$$

$$= P(A) + P(B) + P(A\bar{B})$$

$$P(A) - P(A).P(B) = P(A\bar{B})$$

$$P(A)[1-P(B)] = P(A\bar{B})$$

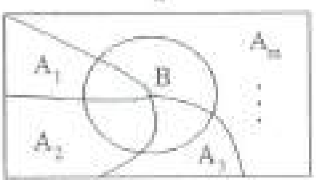
$$P(A).P(\bar{B}) = P(A\bar{B})$$

1.5 BAYE'S THEOREM

Statement :

If $A_1, A_2 \ldots A_m$ are m-mutually exclusive and exhaustive events then

$$P(A_i/B) = \frac{P(A_i).P(B/A_i)}{\sum_{i=1}^{m} P(A_i).P(B/A_i)}$$



Proof :

$$S = A_1 \cup A_2 \cup \ldots \cup A_m$$

$$P(S) = P(A_1 \cup A_2 \cup \ldots \cup A_m)$$

$$= P(A_1) + P(A_2) + \ldots + P(A_3)$$

$$B = A_1 \cup A_2 \cup \ldots \cup A_m$$

$$P(B) = P(A_1B \cup A_2B \cup \ldots \cup A_mB)$$

$$= P(A_1B) + P(A_2B) + \ldots + P(A_mB)$$

$$= P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \ldots + P(A_m).P(B/A_m)$$

$$P(B) = \sum_{i=1}^{m} P(A_i).P(B/A_i) \qquad \ldots(1)$$

We know that

$$P(B).P(A/B) = P(AB)$$

$$P(A).P(B/A) = P(AB)$$

$$\left\{ \begin{array}{l} P(B/A) = \dfrac{P(AB)}{P(A)} \\ P(A).P(B/A) = P(AB) \end{array} \right.$$

$$P(B).\ P(A/B) = P(A).\ P(B/A)$$

$$P(A/B) = \frac{P(A).\ P(B/A)}{P(B)}$$

Let us replace A by A_i

$$P(A_i/B) = \frac{P(A_i).P(B/A_i)}{\sum_{i=1}^{m} P(A_i).P(B/A_i)}$$

PROBLEMS BASED ON BAYE'S THEOREM

Example : 1 ***In a bolt factory machines A_1, A_2, A_3 manufacture respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A_1, A_2, A_3***

Solution :

Let A_i be the probability of manufacturing the bolts.

Let B be probability of defective bolts.

$P(A_i)$	$P(B/A_i)$	$P(A_i)P(B/A_i)$
$P(A_1) = 0.25$	0.05	0.0125
$P(A_2) = 0.35$	0.04	0.014
$P(A_3) = 0.40$	0.02	0.008
$\sum_{i=1}^{m} P(A_i).\ P(B/A_i) = 0.0345$		

P(defective bolt manufactured by machine A_1) = $P(A_1/B)$

$$= \frac{P(A_1).P(B/A_1)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$$

$$= \frac{0.0125}{0.0345} = 0.3623$$

P(defective bolt manufactured by machine A_2) = $P(A_2/B)$

$$= \frac{P(A_2).P(B/A_2)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$$

$$= \frac{0.014}{0.0345} = 0.405$$

P(defective bolt manufactured by machine A_3) = $P(A_3/B)$

$$= \frac{P(A_3).P(B/A_3)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$$

$$= \frac{0.008}{0.0345} = 0.231$$

Example : 2 ***The first bag contains 3 white balls, 2 red balls and 4 black balls. Second bag contains 2 white, 3 red, 5 black balls and third bag contain 3 white, 4 red and 2 black balls. One bag is chosen at random and from it 3 balls are drawn. Out of three balls two balls are white and one is red. What are the probabilities that they were taken from first bag, second bag, third bag.***

Solution :

1^{st} bag	2^{nd} bag	3^{rd} bag
3 White	2 White	3 White
2 Red	3 Red	4 Red
4 black	5 black	2 black

Let A_1, A_2, A_3 denote the events that the bag I, II and III is chosen respectively and Let B be the event that the three balls taken from the selected bag are 2 white and 1 is red.

Selectivity of 2 white and 1 red is an independent event.

Let P(Selecting a bag) = $P(A_i) = {}^1/_3$.

$P(A_i)$	$P(B/A_i)$	$P(A_i)P(B/A_i)$
$P(A_1) = 1/3$	$\frac{3C_2 \times 2C_1}{9C_3} = \frac{6}{84}$	0.0238
$P(A_2) = 1/3$	$\frac{2C_2 \times 3C_1}{10C_3} = \frac{3}{120}$	0.0032
$P(A_3) = 1/3$	$\frac{3C_2 \times 4C_1}{9C_3} = \frac{12}{84}$	0.0476
$\sum_{i=1}^{3} P(A_i).P(B/A_i) = 0.0746$		

We know that

$$P(A_i/B) = \frac{P(A_i).P(B/A_i)}{\sum_{i=1}^{m} P(A_i).P(B/A_i)}$$

(i) Pr(balls selected from 1[st] bag) = $P(A_1/B)$ $= \frac{P(A_1).P(B/A_1)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$

$$= \frac{0.0238}{0.0746} = 0.319$$

(ii) Pr(balls selected from 2[nd] bag) = $P(A_2/B)$ $= \frac{P(A_2).P(B/A_2)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$

$$= \frac{0.0032}{0.0746} = 0.0428$$

(iii) Pr(balls selected from 3[rd] bag) = $P(A_3/B)$ $= \frac{P(A_3).P(B/A_3)}{\sum_{i=1}^{3} P(A_i).P(B/A_i)}$

$$= \frac{0.0476}{0.0746} = 0.638$$

SIMPLE PROBLEMS

1 ***What is the chance that a leap year selected at random will contain 53 Sundays?***

Solution :

A leap year consists of 366 days. Out of these 366 days we have 52 full weeks and 2 days extra.

These two days may be.

(i) Monday	Tuesday
(ii) Tuesday	Wednesday
(iii) Wednesday	Thursday
(iv) Thursday	Friday
(v) Friday	Saturday
(vi) Saturday	Sunday
(vii) Sunday	Monday

of these 7 cases the last two cases contain Sunday and hence we have 2 favourable cases.

$$\therefore \text{Probability} = \frac{\text{number of favourable cases}}{\text{number of exhaustive events}} = \frac{2}{7}$$

2 ***A bag contains 6 red and 7 black balls. Find the probability of drawing a red ball.***

Solution :

$$\therefore \text{Probability} = \frac{\text{number of favourable events}}{\text{number of exhaustive events}} = \frac{6}{36} = \frac{1}{6}$$

3 ***The probability of solving a problem by Asha is $^2/_3$ and that the probability of solving the problem by both Asha and Sophia is $^{14}/_{25}$. The probability of solving either by Asha or Sophia is $^4/_5$. What is the probability of solving the problem by Sophia?***

Solution :

Let us take the probability of solving the problem by Asha is P(A) and by Sophia is P(B).

Given, $P(A) = \frac{2}{3}$, $P(A \text{ or } B) = \frac{4}{5}$, $P(A \text{ and } B) = \frac{14}{25}$

By addition theorem,

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

$$\frac{4}{5} = \frac{2}{3} + P(B) - \frac{14}{25}$$

$$\therefore \quad P(B) = \frac{4}{5} - \frac{2}{3} + \frac{14}{25}$$

$$= \frac{52}{75}$$

The probability of solving the problem by Sophia is $= \frac{52}{75}$

4 ***Sakthi is known to hit the target in 3 out of 4 shots, where as Remya is known hit the target in 2 out of 3 shots. Find the probability of the target being hit wh both person try.***

Solution :

Given A and B be the event of hitting the target by Sakthi and Remya respectivel

Probability when Sakthi hits the target $= \frac{3}{4}$. i.e $P(A) = \frac{3}{4}$

Probability when Remya hits the target $= \frac{2}{3}$. i.e $P(B) = \frac{2}{3}$

A and B are mutually exclusive events.

Then

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

$$= P(A) + P(B) - P(A).P(B)$$

$$= \left(\frac{3}{4} + \frac{2}{3}\right) - \left(\frac{3}{4} \times \frac{2}{3}\right)$$

$$= 0.917$$

Probability of hitting the target when they both try = 0.917.

5 ***A bag contains 10 white and 4 red balls. Two balls are drawn in succession at random. What is the probability that one is white and other is red?***

Solution :

Total number of balls = 14

Drawing 1 white from 10 white balls = $10C_1$

Similarly drawing 1 red from 4 red balls = $4C_1$

Selecting 1 white in the 1^{st} Drawing = $\frac{10C_1}{14C_1}$

Similarly Selecting a red in the 2^{nd} Drawing = $\frac{4C_1}{14C_1}$

$\therefore$ Probability of getting, white and 1 red = $\frac{10C_1}{14C_1} \times \frac{4C_1}{14C_1} = \frac{10}{49}$

6 ***Write the sample space for throwing a three coins at a time. Find the probability of getting (i) Exactly one head (ii) atleast one head (iii) atleast two head.***

Solution :

S = {HHH, TTT, HHT, TTH, THT, THT, HTH, HTT, THH} = 8

(i) Pr{exactly one head} = $\frac{\{TTH, THT, HTT\}}{8} = \frac{3}{8}$

(ii) Pr{atleast one head} = $\frac{\{HHH, HHT, TTH, THT, HTH, THH, HTT\}}{8} = \frac{7}{8}$

(iii) Pr{atleast two head} = $\frac{\{HHH, HHT, HTH, THT\}}{8} = \frac{4}{8} = \frac{1}{2}$

7 ***Two dice are rolled simultaneously, what is the chance that (i) the sum of outcome is 7 (ii) > 10 (iii) < 10.***

Solution :

S = {(1, 1), (1, 2), (1, 3) (1, 4), (1, 5) (1, 6)

(2, 1), (2, 2), (2, 3) (2, 4), (2, 5) (2, 6)

(3, 1), (3, 2), (3, 3) (3, 4), (3, 5) (3, 6)

(4, 1), (4, 2), (4, 3) (4, 4), (4, 5) (4, 6)

(5, 1), (5, 2), (5, 3) (5, 4), (5, 5) (5, 6)

(6, 1), (6, 2), (6, 3) (6, 4), (6, 5) (6, 6)}

(i) Pr {The sum of out come is 7) = $\frac{6}{36} = \frac{1}{6}$

(ii) Pr {Greater than 10) = $\frac{3}{36} = \frac{1}{12}$

(iii) Pr {Less than 10 (or) < 10) = $\frac{30}{36} = \frac{5}{6}$

1.6 RANDOM VARIABLE

Definition

A random variable is single real valued function defined on the sample space.

Eg : In tossing a Coin the outcome head may be assigned the value '1' and the outcome tail may be assigned the value '0'.

1.6.1 Discrete Random Variable

A random variable which takes on a finite or countably infinite number of values is called a discrete random variable.

Probability Distribution or Probability Mass Function (p.m.f)

Let X be discrete random variable and which takes the values $X_1, X_2 \ldots X_n$

Such that $P[X = x_1] = P[X = x_2] = \ldots P_n$

The function P(x) is called probability function p.m.f.

If it satisfies the following conditions

(i) $P(X = x) > 0$ (ii) $\Sigma P(x) = 1$

Distribution Function of a Discrete Random Variable

Let X takes values $x_1, x_2 \ldots x_n$ with probabilities $p_1, p_2 \ldots p_n$ and Let $x_1 < x_2 < \ldots < x_n$. The distribution function F(x) of a random variable X defined in $(-\infty, \infty)$ as $F(x) = P[X \leq x]$.

Properties of Distribution Function

(i) $0 \leq F(x) = 1$

(ii) if $x < y$, then $F(x) \leq F(y)$

(iii) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

(iv) $F(\infty) = \lim_{x \to \infty} F(x) = 1$.

Example : *A random variable 'x' has the following probability function.*

x :	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
P(n) :	*a*	*3a*	*5a*	*7a*	*9a*	*11a*	*13a*	*15a*	*17a*

(i) Determine the value of a.

(ii) Find $P(X<3)$, $P(X\geq3)$, $P(0<X<5)$

(iii) Find the distribution function of α

(i) we know that

$$\sum_{i=0}^{\infty} P(x_i) = 1$$

i.e., $a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$$81a = 1$$

$$a = \frac{1}{81}$$

(ii) $P(X<3) = P(0) + P(1) + P(2)$

$$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{9}{81}$$

$$P(X\geq3) = 1 - P(X<3) = 1 - \frac{9}{81} = \frac{72}{81}$$

(iii) $P(0<X<5) = P(1) + P(2) + P(3) + P(4)$

$= 3a + 5a + 7a + 9a$

$= \frac{3}{81} + \frac{5}{81} + \frac{7}{81} + \frac{9}{81}$

$= \frac{24}{81}$

To find the distribution

x $F(x) = P(X \leq x)$

0	a	$\frac{1}{81}$
1	a + 3a = 4a	$\frac{4}{81}$
2	4a + 5a = 9a	$\frac{9}{81}$
3	9a + 7a = 16a	$\frac{16}{81}$
4	16a + 9a = 25a	$\frac{25}{81}$
5	25a + 11a = 36a	$\frac{36}{81}$
6	36a + 13a = 49a	$\frac{49}{81}$
7	49a + 15a = 64a	$\frac{64}{81}$
8	64a + 17a = 81a	1

1.6.2 Continuous Random Variable

If x is a random variable which can take all values in a interval, then x is called continuous random variable.

Eg : Age, height, weight etc.

Probability Density Function

If x is a random variable such that

$P(x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx) = f(x)\,dx$ then f(x) is called the probability density function, and f(x) satisfied the following properties.

(a) $f(x) > 0 \; \forall \; x$.

(b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cumulative distribution function of a continuous random variable

If X is a continuous random variable, the cumulative distribution function of X is given by

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(x)\,dx.$$

Properties of cumulative distribution functions

(a) F(x) is a non-decreasing function

(b) $F(-\infty) = 0 = \lim_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx$

(c) $F(\infty) = 1 = \lim_{x \to \infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)dx = 1$

(d) $P[a \leq x \leq b] = F(b) - F(a)$

(e) $\frac{d}{dx} F(x) = f(x)$ at all points where F(x) in differential.

Example ***A random variable X has the density function $f(x) = kx$ for $2 \leq x \leq 5$. Find the distribution F(x) and the $P(1 < x < 4)$***

Solution :

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{2}^{5} f(x)\,dx = 1$$

$$\int_2^5 k\, x dx = 1$$

$$k\left(\frac{x^2}{2}\right)_2^5 = 1$$

$$k\left(\frac{25}{2} - \frac{4}{2}\right) = 1$$

$$k = \frac{2}{21}$$

The distribution function is

$$F(x) = \frac{2}{21}\int_2^x x dx$$

$$= \frac{2}{21}\left(\frac{x^2}{2}\right)_2^x$$

$$= \frac{2}{21}\left(\frac{x^2}{2} - 2\right)$$

$$F(x) = 0 \text{ for } x < 2$$

$$= \frac{2}{21}\ (x \leq 4)\ \text{ for } 2 \leq x \leq 5$$

$$= 1 \quad x \geq 5$$

$$P(1 < x < 4) = \frac{2}{21}\int_1^4 x dx$$

$$= \frac{2}{21}\left(\frac{x^2}{4}\right)_1^4$$

$$= \frac{2}{21}\left(8 - \frac{1}{2}\right)$$

$$= \frac{2}{21} \times \frac{15}{21} = \frac{5}{7}$$

1.7 BINOMIAL DISTRIBUTION

Assumptions :

- The random experiment corresponds to only two possible out comes.
- The number of trials is finite.
- The trials are independent.
- The probability of success is a constant from trial to trial.

Probability Mass Function :

A discrete random variable X is said to follow Binomial distribution of it's probability mass function is

$$P(x) = n_{cx}\, p^x q^{n-k},\ x = 0, 1, 2, \ldots n. \qquad \text{here } p + q = 1$$

Properties of Binomial Distribution

- Binomial distribution has two parameters n and p(or q).
- mean = np
- variance = npq
- Standard deviation = $\sqrt{npq}$
- Skewness $(\beta_1) = \dfrac{(q-p)^2}{npq}$
- Kurtosis $(\beta_2) = 3 + \dfrac{1-6pq}{npq}$
- Binomial distribution is symmetrical if p = q = 0.5.
- It is positively skewed if p < 0.5 and it is negatively skewed if p > 0.5.

SIMPLE PROBLEMS

Example : 1 ***The mean and variable of binomial distribution are 6 and 2 respectively. Find $P(X \geq 1)$.***

Solution :

Given that mean = 6 i.e., np = 6 . . . (1)

Variance = 2 i.e. npq = 2 . . . (2)

(2) ÷ (1) ⇒

$$\frac{npq}{np} = \frac{2}{6} \Rightarrow q = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

we know that

$$np = 6$$

$$n = \frac{6}{2} \times 3$$

$$\boxed{n = 9}$$

$$P(X) = n_{C_X}\, p^x\, q^{n-x}$$

$$= 9_{C_X}\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}$$

$$P(X \geq 1) = 1\ P[X<1]$$

$$= 1 - P[X=0]$$

$$= 1 - 9_{C_0}\left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{9-0}$$

$$= 1$$

Example : 2 ***The mean of a binomial distribution is 20 and standard deviation is 4. Find out n, p, and q.***

Solution :

Mean = np = 20

Stansdard deviation = $\sqrt{npq} = 4$

$$npq = 16$$

$$q = \frac{npq}{np} = \frac{16}{20} = 0.8$$

$$p = 1 - 0.8 = 0.2$$

$$n = \frac{20}{0.2} = 0.2$$

Example : 3 ***The probability of a defective bolt is 0.2. Find (a) the mean and S.D for the distribution of defective bolts in a total of 1000 and (b) find the co-efficient of skewness and kurtosis.***

Solution :

$p = 0.2, q = 0.8, n = 1000$

mean $= np = 1000 \times 0.2 = 200$

$$S.D = \sqrt{npq} = \sqrt{1000 \times 0.2 \times 8} = 12.6$$

$$\beta_1 = \frac{(q-p)^2}{npq} = \frac{(0.8 - 0.2)^2}{1000 \times 0.2 \times 0.8} = \frac{0.36}{160} = 0.225$$

$$\beta_2 = 3 + \frac{(1-6pq)}{npq} = 3 + \left[\frac{1-(6 \times 0.2 \times 0.8)}{1000 \times 0.2 \times 0.8}\right]$$

$$= 3.0025$$

Example : 4 ***The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students (a) none, (b) one, and (c) atleast one will graduate.***

Solution :

$n = 5, p = 0.4, q = 0.6$

$$P(X) = 5_{C_X}\, p^x\, q^{n-x}$$

(a) The probability of zero success

$$= 5_{C_0} \left(\frac{2}{10}\right)^0 \left(\frac{6}{10}\right)^5$$

$$= 0.078$$

(b) The probability of one success

$$= 5_{C_1} \left(\frac{4}{10}\right)^1 \left(\frac{6}{10}\right)^4$$

$$= 0.259$$

(c) The probability of atleast one success

= 1 − probability of number of success

= 1 − 0.078

= 0.922

1.7.1 FITTING OF BINOMIAL DISTRIBUTION

Example : 5 *The following data show the number of seeds germinating out of 10 on damp for 80 set of seeds. Fit a binomial distribution to this data.*

x :	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
y :	*6*	*20*	*28*	*12*	*8*	*6*	*0*	*0*	*0*	*0*	*0*

x	f	fx
0	6	0
1	20	20
2	28	56
3	12	36
4	8	32
5	6	30
6	0	0
7	0	0
8	0	0
9	0	0
10	0	0
	$N = \Sigma f = 80$	$\Sigma fx = 174$

$$\text{Mean} = x = \frac{\Sigma fx}{N} = \frac{174}{80} = 2.175$$

$$\text{Mean} = np = 2.175$$

$$\Rightarrow \quad p = \frac{2.175}{10} \qquad \text{where } n = 10$$

$$p = 0.2175$$

$q = 0.2175$

$= 0.7825$

The biniomial distribution to be fitted to the data is $N_x n_{C_x}\ p^x\ q^{n-x}$

X	$80 \times 10C_x\ (0.2175)^x\ (0.7825)^{10-x}$	Expected value
0	$80 \times 10C_0\ (0.2175)^0\ (0.7825)^{10-0}$	6.9
1	$80 \times 10C_1\ (0.2175)^1\ (0.7825)^{10-1}$	19.1
2	$80 \times 10C_2\ (0.2175)^2\ (0.7825)^{10-2}$	24.0
3	$80 \times 10C_3\ (0.2175)^3\ (0.7825)^{10-3}$	17.9
4	$80 \times 10C_4\ (0.2175)^4\ (0.7825)^{10-4}$	8.6
5	$80 \times 10C_5\ (0.2175)^5\ (0.7825)^{10-5}$	2.9
6	$80 \times 10C_6\ (0.2175)^6\ (0.7825)^{10-4}$	0.7
7	$80 \times 10C_7\ (0{:}2175)^7\ (0.7825)^{10-3}$	0.1
8	$80 \times 10C_8\ (0.2175)^8\ (0.7825)^{10-2}$	0
9	$80 \times 10C_9\ (0.2175)^9\ (0.7825)^{10-1}$	0
10	$80 \times 10C_{10}\ (0.2175)^{10}\ (0.7825)^{10}$	0
	Total	80.1

Example : 5 *Eight coins are tossed at a time 256 times. Number of heads observed at each throw is recorded and the results are given below. Find the expected frequencies. What are the theoretical values of mean and standard deviation? Calculate also the mean and S.D of the observed frequencies.*

No. of heads :	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Frequency :	*2*	*6*	*30*	*52*	*67*	*56*	*32*	*10*	*1*

Solution :

The chance of getting a head in a single throw of one coin is ½. Hence p = ½, q = ½, n = 8, N = 256.

X	$N_x n_{C_x} p^x q^{n-x}$	Expected value
0	$256 \times 8C_0 (½)^0 (½)^{8-0}$	1
1	$256 \times 8C_1 (½)^1 (½)^{8-1}$	8
2	$256 \times 8C_2 (½)^{20} (½)^{8-2}$	28
3	$256 \times 8C_3 (½)^3 (½)^{8-3}$	56
4	$256 \times 8C_4 (½)^4 (½)^{8-4}$	70
5	$256 \times 8C_5 (½)^5 (½)^{8-5}$	56
6	$256 \times 8C_6 (½)^6 (½)^{8-6}$	28
7	$256 \times 8C_7 (½)^7 (½)^{8-7}$	8
8	$256 \times 8C_8 (½)^8 (½)^{8-8}$	1
	Total	256

The mean of the above distribution is np = 8 × ½ = 4

The standard deviation is $\sqrt{npq} = \sqrt{½ \times ½ \times 8}$

$$= \sqrt{2}$$

$$= 1.414$$

1.8 POISSON DISTRIBUTION

The distribution describes the behaviour of rate events and has back known as the law of improbable events. It was found by French mathematician Simeon D.Poisson, poisson distribution can study when we know the mean value of occurrences of an event without knowing the sample space. Mathematically, the poisson distribution is limiting form of Binomial distribution as n tends to infinity and p approaches zero such that np = m remain constant.

Probability Mass Function :

A discrete random variable x is said to follow poisson distribution, If its probability density function is

$$\lambda(x) = \frac{e^{-\lambda}\lambda^{x}}{x\,!} \qquad x = 0, 1, 2, \ldots \infty$$

1. Mean = λ
2. Variance = λ
3. Standard deviation = $\sqrt{\lambda}$
4. Skewness given by $\beta_1 = \frac{1}{m}$
5. Kurtosis, given by $\beta_2 = 3 + \frac{1}{m}$

Example : 1 *If x is poisson variate such that P(x=1) = and P(x=2) = , find P(x=0) and P(x=3).*

$$P(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x\,!}$$

$$P(x=1) = \frac{e^{-\lambda}\lambda^{1}}{1\,!} = \frac{3}{10} \qquad \ldots(1)$$

$$P(x=2) = \frac{e^{-\lambda}\lambda^{2}}{2\,!} = \frac{1}{5} \qquad \ldots(2)$$

$$(1) \Rightarrow \quad e^{-\lambda}\lambda = \frac{3}{10} \qquad \ldots(3)$$

$$(2) \Rightarrow \quad e^{-\lambda}\lambda^{2} = \frac{2}{5} \qquad \ldots(4)$$

$$\frac{(3)}{(4)} = \frac{1}{\lambda} = \frac{\frac{3}{10}}{\frac{2}{5}} = \frac{3}{4}$$

$$\lambda = \frac{4}{5}$$

$$P(x=0) = \frac{e^{-\lambda}\lambda^0}{0\,!} = e^{-4/3}$$

$$P(x=3) = \frac{e^{-\lambda}\lambda^3}{3\,!} = \frac{e^{-4/3}\left(\frac{4}{3}\right)^3}{3\,!}$$

Example : 2 *If the mean of a poisson distribution is 4, find (i) S.D (ii) β_1 (iii) β_2*

1. Mean = $\lambda = 4$
2. Standard Deviation = $\sqrt{\lambda} = \sqrt{4} = 2$
3. Skewness = $\beta_1 = \frac{1}{\lambda} = \frac{1}{4} = 0.25$
4. Kurtosis = $\beta_2 = 3 + \frac{1}{\lambda} = 3 + \frac{1}{4} = 3.25$

FITTING OF POISSON DISTRIBUTION

Example : 3 *100 Car Radios are inspected as they come off the production line and number of defects per set is recorded below.*

No. of Defects :	*0*	*1*	*2*	*3*	*4*
No. of Sets :	*79*	*18*	*2*	*1*	*0*

Fit a poisson distribution to the above data and calculate the frequencies of 0, 1, 2, 3 and 4 defects.

No. of defects x	No. of sets f	fx
0	79	0
1	18	18
2	2	4
3	1	3
4	0	0
	N = 100	$\Sigma fx = 25$

$$\lambda = x = \frac{\Sigma fx}{N} = \frac{25}{100}$$

$$= 0.25$$

$$e^{-0.25} = 0.779$$

X	$N = \frac{e^{-\lambda}\lambda^x}{x!}$	**Expected value**
0	$100 \times \frac{e^{-0.25}(0.25)^0}{0!}$	77.90
1	$100 \times \frac{e^{-0.25}(0.25)^1}{1!}$	19.48
2	$100 \times \frac{e^{-0.25}(0.25)^2}{2!}$	2.44
3	$100 \times \frac{e^{-0.25}(0.25)^3}{3!}$	0.20
4	$100 \times \frac{e^{-0.25}(0.25)^4}{4!}$	0.20
	Total	$\simeq$ 100

Example : 4 *Fit a Poisson Distribution to the following*

x :	*0*	*1*	*2*	*3*	*4*
y :	*123*	*59*	*14*	*3*	*1*

x	f	fx
0	123	0
1	59	59
2	14	28
3	3	9
4	1	4
	$N = \Sigma f = 200$	$\Sigma fx = 100$

$$\lambda = \bar{x} = \frac{\Sigma fx}{N} = \frac{100}{200} = 0.5$$

Calculation of expected frequencies

x	NxP(x)	Expected value
0	123	121
1	59	61
2	14	15
3	3	3
4	1	6
		200

1.8.1 DIFFERENCE BETWEEN BINOMIAL AND POISSON DISTRIBUTION

Binomial Distribution	Poisson Distribution
1. $P(x) = N_x n_{C_x} p^x q^{n-x}$, x = 0, 1, 2 . . . where q = 1 – p	$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2 . . . here λ = n p
2. The Binomial Distribution is characterised by two parameters p, n	The Poisson Distribution is characterised by a single parameter λ.
3. The sample space for Binomial Distribution is {0, 1, 2, . . . n}.	The sample space for Poisson Distribution is {0, 1, . . . n}.
4. Mean = np, variance = npq $\mu_3 = npq(q-p)$, $\mu_4 = npq[1-6pq-3npq]$	Mean = λ, variance = λ $\mu_3 = m$ $\mu_4 = 3m^2 + m$
5. Binomial frequency distribution $f(x) = Nn_{C_x} p^x q^{n-x}$, x = 0, 1, 2, . . . n	Poisson frequency distribution $f(x) = \frac{Ne^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2, . . . n
6. Discovered by Swiss Mathematician James Bernoulli through it was published in 1713.	Introduced by French Mathematician S.D. Poisson in 1837.
7. If n is small we use Binomial distribution.	If n is large close to zero we use Poisson distribution.

1.9 NORMAL DISTRIBUTION

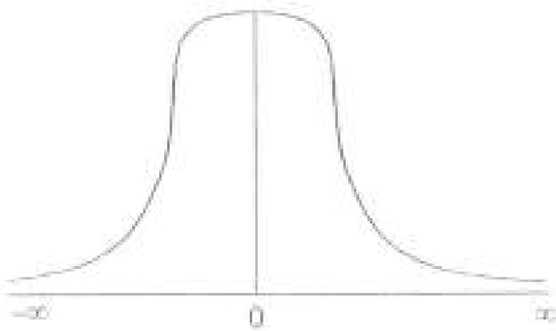
The Normal distribution was introduced by the French Mathematician Abraham De Moivre in 1733. Demoivre, who used this distribution to approximate probabilities connected with coin tossing called if the exponential bell shaped curve.

Normal Distribution :

The Normal distribution is a continuous distribution given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

1.9.1 CHARACTERISTICS OF THE NORMAL DISTRIBUTION



- ✓ The curve in symmetrical. It is bell shaped curve.
- ✓ The value of mean, median and mode, will coincide because the distribution is symmetrical and single peaked.

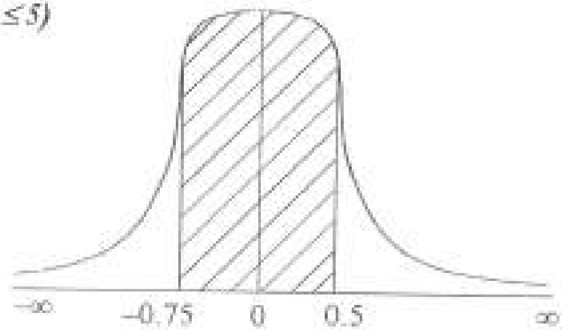
  i.e., Mean = Median = Mode
- ✓ The normal distribution is a two parameter probability distribution. The parameters mean and standard deviation (μ, σ) completely determine the distribution.
- ✓ Since the distribution is symmetrical the moment of co-efficient of skewness $\beta_1 = 0$, $\beta_2 = 3$ (Mesokurtic curve).
- ✓ The curve has a single peak point (i.e) the distribution is unimodel.
- ✓ The maximum ordinate is at $x = \mu$. Its value is $\frac{1}{\sigma\sqrt{2\pi}}$. It is important to note that when the S.D increases, the maximum ordinate decreases and vice versa.

Importance of Normal Distribution

- ❖ Most of the distribution occurring in practice. (e.g) Binomial, Poisson, etc. Can be approximated by normal distribution. Moreover, many of the sampling distribution (e.g) Student; 't' Snedecor's F, Chiv Square distributions, etc. tend to normality for large samples.

- Even if a variable is not normally distributed, it can sometimes be brought to normal form by simple transformations of variable.
- Many of the distribution of sample statistic (e.g the distribution of sample mean, sample variance etc.) tend to normality for large samples and as such they can best to studied with the help of the normal curve.
- The central limit theorem of the normal distribution is the most important as it enables us to draw inferences about the universe by making sample studies. If a random sample is taken from universe, then as the sample size increases the mean of the sample approaches the normal distribution.
- Normal distribution also finds considerable application in the theory of Statistical Quality Control.
- The basis of test of significance is mainly upon the fundamental assumption that the population from which the sample have been drawn in normally distributed.

Example : 1 ***If X is normally distributed with mean 8 and S.D = 4 find (a) $P(5 \le x \le 10)$ (b) $P(x \ge 15)$, (c) $P(x \le 5)$***



Solution :

$\mu = 8, \sigma = 4$

The standard normal variate is

$$Z = \frac{x - \mu}{\sigma}$$

(a) $P(5 \le x \le 10)$

$$\text{when } x = 5, Z = \frac{5-8}{4} = \frac{-3}{4} = -0.75$$

$$\text{when } x = 10, Z = \frac{10-8}{4} = \frac{2}{4} = 0.5$$

$$P(5 \le x \le 10) = P(-0.75 \le Z < 0.5)$$

$$= P(-0.75 \le Z \le 0) + P(0 < Z < 0.5)$$

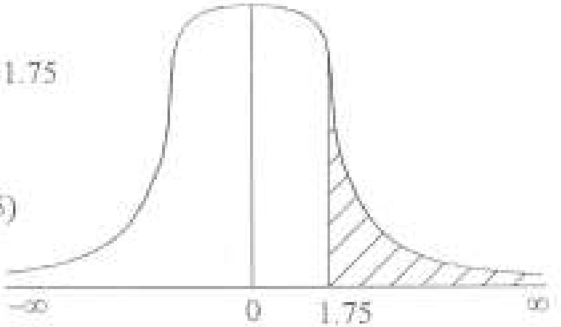
$$= P(0 \le Z < 0.75) + P(0 < Z < 0.5)$$

$= 0.2734 + 0.1915$

$= 0.4649$

(b) P(x ≥15)

when $x = 15,\ Z = \frac{15-8}{4} = \frac{7}{4} = 1.75$



$P(x \geq 15) = P(Z \geq 1.75)$

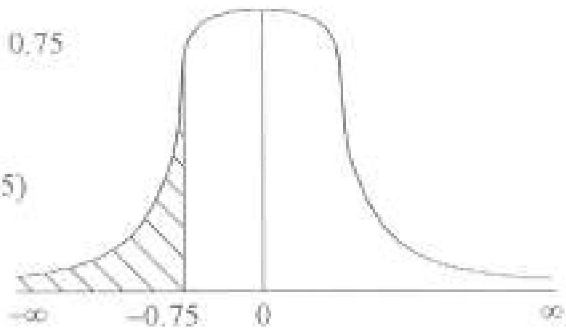
$= 0.5 - P(0 \leq Z \leq 1.75)$

$= 0.5 - 0.4599$

$= 0.0401$

(c) P(x ≤5)

when $x = 5,\ Z = \frac{5-8}{4} = \frac{-3}{4} = -0.75$



$P(x \leq 5) = P(Z \leq -0.75)$

$= 0.5 - P(0 < Z < 0.75)$

$= 0.5 - 0.2734$

$= 0.3266$

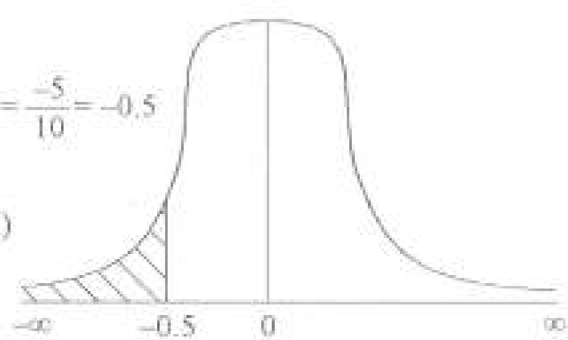
Example : 2 ***The marks obtained in a certain examination follow normal distribution with mean 45 and S.D 10. If 1000 students appeared at the examination. Calculate the number of students scoring.***

(i) Less than 40 marks ***(ii) More than 60 marks***

Solution :

(i) x ≤40

when $x = 40,\ Z = \frac{x-\mu}{\sigma} = \frac{40-45}{10} = \frac{-5}{10} = -0.5$



$P(x \leq 40) = P(Z < -0.5)$

$= 0.5 - P(0 < Z < 0.5)$

$= 0.5 - 0.1915$

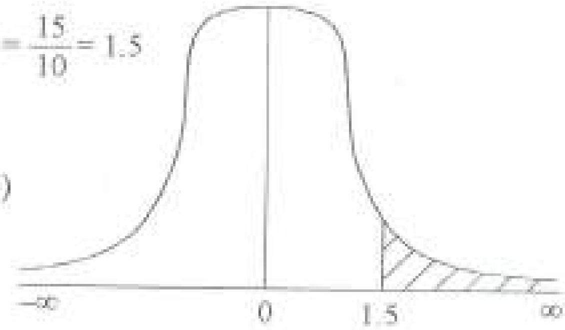
$= 0.3085$

Number of students = 1000 × 0.3085

≈ 309

(ii) For x ≥ 60

when x = 40, $Z = \frac{x-\mu}{\sigma} = \frac{60-45}{10} = \frac{15}{10} = 1.5$



$P(x \geq 60) = P(Z \geq 1.5)$

$= 0.5 - P(0 < Z < 1.5)$

$= 0.5 - 0.4332$

$= 0.0668$

Number of students = 1000 × 0.0688

= 66.8

$\approx 67.$

1.10 UNIFORM OR RECTANGULAR DISTRIBUTION

A random variable X is said to have continuous uniform over an internal (a, b) if its density function f(x) is a constant k, over the entire range of x.

$$f(x) = \begin{cases} k & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

1.10.1 Properties of Uniform Distribution

(i) Mean $= \frac{a+b}{2}$

(ii) Variance $= \frac{(b-a)^2}{12}$

(iii) The M.GF of uniform distribution is $M_X(t) = \frac{1}{b-a}\left[\frac{e^{bt} - e^{at}}{t}\right]$

Example : 1 *If x has a uniform distribution with p.d.f.*

$$f(x) = \begin{cases} 1/10 & 10 < x < 20 \\ 0, & \text{otherwise} \end{cases}$$

Find P(x ≥ 15)

$$P(x \geq 15) = \int_{15}^{20} f(x)\,dx$$

$$= \int_{15}^{20} \frac{1}{10}\,dx$$

$$= \frac{1}{10}\,[x]_{15}^{20}$$

$$= \frac{1}{10}\,[20 - 15]$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}$$

EXERCISE

1. When two dice are tossed. What is the probability of getting four as the sum of the face number? ***[Ans : 1/12]***
2. What is the probability that there will be 53 Sunday in (i) a leap year (ii) a non-leap year? ***[Ans : 2/7, 1/7]***
3. $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{5}$, $P(AB) = \frac{1}{3}$. Find $P(A \cup B)$.
4. A company has four producing units Viz, S_1, S_2, S_3 and S_4 which contribute 30%, 20%, 28% and 22% respectively to the total output. It was observed that these sections respectively produce 1%, 2%, 3% and 4% defective units. If a unit is selected at random and found to be defective, what is the probability that the unit. So selected has come from either section S_1 or section S_4.
5. There are 3 boxes containing respectively 1 white, 2 red, 3 black balls; 2 white, 3 red, 1 black balls; 3 white, 1 red and 2 black balls; A box is chosen at random and from it two balls are drawn at random. The two balls are 1 red and 1 white. White is the probability that they come from (i) the first box (ii) third box.